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IONOSPHERIC HEATING WITH OBLIQUE WAVES

Vol. 1. Electron Density Perturbations

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The transport and chemical equations that describe electron density and temperature changes produced by electric fields in the ionosphere are derived and cast into a form suitable for numerical integration. By using those equations in conjunction with previously developed methods for calculating electric fields near caustics, the density and temperature changes produced by an oblique ionospheric high-frequency heater can be computed straightforwardly. The necessary inputs are: (1) heater power, (2) heater radiation pattern, (3) heater frequency, and (4) model ionosphere. Analytic solutions for idealized heater beamshapes indicate that heat conduction and diffusion are also presented.

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PREFACE

This report extends earlier work* on oblique-angle ionospheric heating to include heat conduction and diffusion. It derives equations suitable for numerical integration, which can be used with previously developed methods to calculate changes in electron temperature and density produced by powerful high-frequency transmitters.

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*Field, E. C., Jr., and C. R. Warber, *Ionospheric Modification with Obliquely Incident Waves: Electron Heating and Parametric Instabilities*, Rome Air Development Center, Interim Report RADC-TR-85-188, ADA162603, October 1985.

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I. INTRODUCTION

During the past 15 years a number of powerful high-frequency (HF) radio transmitters have been developed solely to transmit waves strong enough to produce measurable nonlinear effects in the ionosphere. At first the only goal was to produce a detectable change in electron temperature, so the early tests were called heating experiments. Many types of nonlinear effects were observed, however, and such tests are now called ionospheric modification experiments. Several authors [1-3] have summarized the phenomena observed in modification experiments.

Although some Soviet work [4-7] has used oblique incidence, virtually all modification experiments have transmitted the powerful modifying wave at vertical incidence. Radar and communication systems operate at oblique incidence, however. For reasons given below, an obliquely incident modifying wave might not produce the same effects as a vertically incident one. It is therefore of interest to examine the effects of a strong oblique wave on the ionosphere and on waves that propagate through the modified region.

The nonlinear processes that lead to ionospheric modification are divided into two categories: (1) simple heating, which alters temperature and hence, reaction rates, collision frequencies, and particle densities, and (2) generation of parametric instabilities, which cause a myriad of phenomena, including spread-F and geomagnetic-field-aligned irregularities. Both categories can change the amplitude, phase, or path of a signal that traverses the modified region, and both depend strongly on the incidence angle of the modifying wave.

As the incidence angle is increased, heating, which is proportional to the local power density, is affected by two competing trends. The field is weakened by the increased path length, but strengthened by focusing near caustics [8]. Field and Warber [9, 10] showed that focusing near caustics overcomes the greater geometric spreading suffered by oblique modifying waves, and that intense electric fields can occur. They only calculated electric field strength, however, and did not account for transport processes; nor

did they calculate the electron density changes caused by chemistry changes and diffusion driven by such focused fields. Most parametric instabilities cannot be excited by oblique waves, because only vertical waves can reach heights where matching conditions on frequency are met.

The present report extends the work of Field and Warber to include the effects of heat conduction and diffusion. Those processes are more important for oblique incidence than for vertical incidence [11], because at oblique incidence the intense fields are strongly localized near caustics, which are narrow. The resulting gradients are stronger than those caused by vertical waves, whose energy is deposited over wider ionospheric regions. This report emphasizes the underlying theory and derives equations suitable for numerical calculation. In addition, sample cases are presented that illustrate how heat conduction and ambipolar diffusion affect temperature and density changes in the caustics of intense oblique waves.

II. EQUATIONS GOVERNING ELECTRON HEATING AND TRANSPORT

The equations that govern the changes in electron temperature and density produced by an intense radio wave in the ionosphere are presented in this section. Analytic solutions for idealized situations that are useful for understanding the physical processes are provided.

Some of the equations below are similar to those in Gurevich [1], whose notation we use whenever convenient. Certain lengthy derivations, not given by Gurevich, are outlined in the appendixes.

Because the magnetic pressure greatly exceeds the plasma pressure at altitudes of interest, we consider only field-aligned particle motion and energy transport, ignoring the small cross-field drift and thermal conduction. We use a coordinate system with the x-direction along the geomagnetic field. That coordinate is related to height z by $x \sin \psi = z$, where ψ is the dip angle.

ELECTRON TEMPERATURE

If we neglect ion heating, the electron temperature T is described by the following equation (e.g., Meltz and LeLevier [12]):

$$\frac{dT}{dt} = \frac{\partial}{\partial x} \kappa \frac{\partial T}{\partial x} + \delta \nu_e T_o \left[\frac{E^2}{E_p^2} - \frac{(T - T_o)}{T_o} \right] , \qquad (1)$$

where T_o is the ambient electron temperature, E is the electric field of the modifier wave, δ is the fractional energy lost by an electron per collision, and ν_e the electron collision frequency.

The quantity E_p in Eq. (1) is defined as

$$E_p^2 \approx \frac{3KT_o m \delta \omega^2}{e^2} . \qquad (2)$$

where K is Boltzmann's constant, m is the electron mass, e is the

electron charge, and ω is the modifier wave angular frequency. Because E_p has units of volts per meter, it is often called the characteristic plasma field. The form given by Eq. (2) is valid when the wave frequency greatly exceeds the electron gyrofrequency.

The quantity κ in Eq. (1) is the thermal diffusivity and is given by

$$\kappa \approx A \frac{KT}{m\nu_e} , \tag{3}$$

where A is a constant that depends on the sophistication of the theory used to derive κ. Although A can be as large as 2, it is often assumed to be unity. The thermal diffusivity accounts for heat conduction.

The coefficients κ and ν_e both depend on temperature, so Eq. (1) is non linear. Moreover, the dependence of κ on x prevents Eq. (1) from assuming the form of a canonical diffusion equation. The criterion for linearizing Eq. (1) is that $\Delta T = T - T_o << T$, which is always true if E^2/E_p^2 is small, and, in the presence of strong heat conduction can be true even if E^2/E_p^2 is of order unity. The x-dependence of the thermal diffusivity κ can then be ignored if the gradients in the ambient ionosphere are weaker than those caused by the modifying wave. We make both of those simplifications in the present analysis, and rewrite Eq. (1) in the form

$$\tau_T \frac{d\Delta T}{dt} - L_T^2 \frac{\partial^2 \Delta T}{\partial x^2} = T_o \left(\frac{E^2}{E_p^2} - \frac{\Delta T}{T_o} \right) , \tag{4}$$

where, following Gurevich [1], we have defined

$$\tau_T = (\delta \nu_e)^{-1} \quad s , \tag{5}$$

$$L_T^2 = \tau_T \kappa \quad m^2 , \tag{6}$$

and ν is computed using $T = T_o$. The quantity τ_T is the characteristic time required to achieve most of the temperature change. The quantity

L_T is the characteristic distance over which heat is conducted in time τ_T, and thus measures the dimension of the region over which the energy deposited by the modifying wave is spread through heat conduction.

If we use local height averages for τ_T and L_T^2, which is equivalent to assuming that those coefficients are locally height-independent, Eq. (4) can be solved as shown in Appendix A. The result is:

$$\Delta T(x,t) = \frac{T_o}{\tau_T} \int_o^t ds \int_{-\infty}^{\infty} d\xi \left[\frac{E^2(\xi,s)}{E_p^2}\right] \frac{\exp\left[\frac{-(t-s)}{\tau_T} - \frac{(x-\xi)^2}{4\kappa(t-s)}\right]}{\sqrt{4\pi\kappa(t-s)}} \quad . \tag{7}$$

Once the field E of the modifying wave has been calculated, Eq. (7) can be used to calculate both the time and space dependence of electron temperature.

The steady-state reached by the electron temperature after the initial transient period may be of interest. Appendix B shows that the asymptotic behavior of Eq. (7), applicable to late times where $t >> \tau_T$, is simply

$$\Delta T(x) \xrightarrow[t >> \tau_T]{} \frac{T_o}{2L_T} \int_{-\infty}^{\infty} dx' \left[\frac{E^2(x')}{E_p^2}\right] \exp\left(\frac{-|x' - x|}{L_T}\right) \quad . \tag{8}$$

ELECTRON DENSITY

If we again use locally height-averaged values for ambient ionospheric properties, we can write the equation that governs the electron density in the following form:

$$\frac{\partial N}{\partial t} - D_a \frac{\partial^2 N}{\partial x^2} - \frac{N}{T} D_{Ta} \frac{\partial^2 T}{\partial x^2} = P - \alpha N^2 - \beta N \quad . \tag{9}$$

In Eq. (9), P is the ion-pair production rate, α is the electron-ion recombination coefficient, and β is a loss rate that accounts for

interactions between the ion O^+ and the molecules N_2 and O_2. The ambipolar diffusion coefficient is

$$D_a = \frac{K(T_o + T_i)}{M\nu_i} , \tag{10}$$

and the ambipolar thermal diffusivity is

$$D_{Ta} = \frac{KT_o}{M\nu_i} = R_T D_a , \tag{11}$$

where $R_T = T_o/(T_o + T_i)$. M is the mean ion mass, T_i is the ion temperature, and ν_i is the ion-neutral collision frequency. Equation (9) thus accounts for diffusion caused by excess particle concentrations, diffusion driven by thermal gradients, and density changes brought about through the temperature dependence of reaction rates.

The fractional density change is always small, so we can write

$$N = N_o + n , \tag{12}$$

where N_o is the ambient electron density. We can also use the fact that $n << N_o$ to linearize Eq. (9). The result is:

$$\frac{\partial n}{\partial t} - D_a \frac{\partial^2 n}{\partial x^2} - R_T \frac{N_o}{T} D_a \frac{\partial^2 \Delta T}{\partial x^2} \approx -(2\alpha_o N_o \beta)n - [\alpha(T) - \alpha_o] N_o^2 . \tag{13}$$

where $\alpha_o \equiv \alpha(T_o)$ is the unperturbed recombination coefficient and, because the field E is not strong enough to alter the ambient production rate P, we have used

$$P = \alpha_o N_o^2 + \beta_o N_o . \tag{14}$$

Because β is virtually independent of temperature, we have neglected a term proportional to $\beta - \beta_o$. By defining

$$\tau_N = (2\alpha_o N_o + \beta)^{-1} \ , \tag{15}$$

$$L_N^2 = \tau_N D_a \ , \tag{16}$$

and

$$\gamma = - \frac{\tau_N (\alpha - \alpha_o) NT}{\Delta T} \tag{17a}$$

$$\approx -\tau_N NT \frac{\partial \alpha}{\partial T} \ . \tag{17b}$$

we can rewrite Eq. (13) in a form analogous to Eq. (4):

$$\tau_N \frac{\partial n}{\partial t} - L_N^2 \frac{\partial^2 n}{\partial x^2} - R_T \frac{N_o}{T_o} \frac{\partial^2 \Delta T}{\partial x^2} \approx -n + \gamma \frac{N_o}{T_o} \Delta T \ . \tag{18}$$

We reconcile Eq. (17) with the definition of γ used by Gurevich [1] by noting that the electron loss rate q_r is $q_r = \alpha N^2 + \beta N$, so $\tau_N = (\partial q_r/\partial N)^{-1}$ and $\partial\alpha/\partial T = N^{-2}\partial q_r/\partial T$. The time-dependent solution to Eq. (18) can be written in terms of a Green's function, but that result is too complicated to be useful. Therefore, we restrict attention to the following steady-state solution, which is derived in Appendix C·

$$\frac{\Delta N}{N_o} = \frac{\gamma L_N L_T}{2\left(L_N^2 - L_T^2\right)} \int_{-\infty}^{\infty} dy \left[\frac{E^2(y)}{E_p^2}\right]\left[\frac{\exp\left(-|x - y|/L_N\right)}{L_N} - \frac{\exp\left(-|x - y|/L_T\right)}{L_N}\right]$$

$$- \frac{R_T L_N}{2\left(L_N^2 - L_T^2\right)} \int_{-\infty}^{\infty} dy \left[\frac{E^2(y)}{E_p^2}\right]\left[\frac{\exp\left(-|x - y|/L_N\right)}{L_N} - \frac{\exp\left(-|x - y|/L_T\right)}{L_N}\right] \ .$$

(19)

As we have shown for the temperature change ΔT, once the modifying field E is calculated using a method described by Field and Warber [9], the density change n can be calculated by performing a straightforward numerical integration.

III. VALUES OF TRANSPORT COEFFICIENTS AND ELECTRON LOSS RATES

The coefficients and reaction rates that appear in Eqs. (4) through (8), (18), and (19) are presented in this section. Most of those quantities cannot be specified precisely, because either they depend on ionospheric properties, which fluctuate, or they depend on processes that are poorly understood. Gurevich [1] suggests nominal values; we consider them reasonable and use them as inputs in our analysis. The following discussion gives the rationale for using those values, and indicates the uncertainties. We limit discussion to altitudes between 200 km and 300 km, because that is where the most important heating occurs.

MODEL IONOSPHERE

Table 1 summarizes the ionospheric parameters suggested by Gurevich [1]. Those parameters represent nominal values suitable for examples, but their values vary considerably, depending on sunspot activity, latitude, and local time. In our notation, N_{N_2} denotes the number density of nitrogen molecules, N_{NO^+} denotes the density of nitric oxide ions, and so forth. Oxygen atoms are the main neutral constituent at the altitudes shown.

RESPONSE TIMES, TRANSPORT COEFFICIENTS, AND CHARACTERISTIC LENGTHS

Nominal values for response times τ_T and τ_N, transport coefficients κ and D_a, and characteristic lengths L_T and L_N are derived by inserting parameter values from Table 1 into formulas given in Sec. II.* The resulting values are listed in Table 2. Although the

---

*No simple formula is given for the energy loss fraction δ, because different processes contribute to δ at different altitudes. At altitudes so high ($\geq$ 250 km) that elastic collisions between electrons and ions predominate, δ is simply $2m/M$. At lower altitudes, where inelastic collisions with molecules are important, δ must be calculated from the appropriate energy-dependent cross sections. Gurevich [1] gives altitude-dependent recipes for calculating δ.

TABLE 1. Model Ionosphere.

z(km)	N_e (cm^{-3})	T_O (°k)	T_i (°k)	ν_e (S^{-1})	ν_i (S^{-1})	M(u)	N_{NO^+}/N_O	$N_{O_2^+}/N_O$	N_{O^+}/N_O	N_{N_2}	N_{O_2}
Day											
200	5×10^5	1300	1100	590	5.2	16	0.045	0.045	0.09	4.8×10^9	5.9×10^8
250	10^6	1700	1300	700	1.8	16	6×10^{-3}	5×10^{-3}	0.98	1.1×10^9	1.2×10^8
300	1.6×10^6	2000	1400	810	0.75	16	0	0	0.99	3.2×10^8	2.7×10^7
Night											
200	3×10^3	900	850	105	5.2	16	0.045	0.045	0.09	4.7×10^9	5.6×10^8
250	10^4	1000	910	50	1.8	16	6×10^{-3}	5×10^{-3}	0.98	7.7×10^8	7.1×10^7
300	10^5	1200	930	122	0.75	16	0	0	0.99	1.4×10^8	1.0×10^7

SOURCE: [1].

TABLE 2. Transport and Loss-Rate Coefficients.

z(km)	τ_T(s)	τ_N(s)	κ(km^2/s)	D_a(km^2/s)	L_T(km)	L_N(km)	R_T	α_O(cm^6/s)	β(cm^3/s)	γ
Day										
200	5.2	32	56	0.17	17	2.3	0.54	7.4×10^{-9}	1.7×10^{-2}	0.08
250	9.1	140	61	0.6	25	9.2	0.57	7×10^{-10}	3.5×10^{-3}	0
300	14.2	490	61	1.60	28	28	0.59	0	8.6×10^{-4}	0
Night										
200	6.0	60	120	0.15	31	3	0.51	1.1×10^{-8}	1.6×10^{-2}	0
250	25.0	400	370	0.61	103	16	0.52	1.2×10^{-9}	2.2×10^{-3}	0
300	71.0	2200	150	2.4	99	72	0.56	0	3.4×10^{-4}	0

SOURCE: [1].

nominal values given are useful for discussion, different ones must be defined to correspond to the actual model ionosphere used for a specific calculation.

ELECTRON LOSS RATES

Gurevich [1] suggests the following formulas for the loss rates α and β:

$$\alpha = \alpha_1 \left(\frac{N_{NO^+}}{N_O}\right) + \alpha_2 \left(\frac{N_{O_2^+}}{N_O}\right) , \tag{20}$$

where

$$\alpha_1 = 5 \times 10^{-7} \left(\frac{300}{T}\right)^{1.2} . \tag{21a}$$

$$\alpha_2 = 2.2 \times 10^{-7} \left(\frac{300}{T}\right)^{0.7} . \tag{21b}$$

and

$$\beta = 10^{-12} N_{N_2} + 2 \times 10^{-11} N_{O_2} . \tag{21c}$$

Other authors give somewhat different values for the above loss rates. For example, McEwan and Phillips [13] indicate that α_1 behaves as $(300/T)^{1.0}$, whereas Bates [14] uses the relationships

$$\alpha_1 = 4.2 \times 10^{-7} \left(\frac{300}{T}\right)^{\lambda_1} . \tag{22}$$

$$\alpha_2 = 1.6 \times 10^{-7} \left(\frac{300}{T}\right)^{\lambda_2} , \tag{23}$$

where $0.37 \le \lambda_1 \le 0.9$ and $0.55 \le \lambda_2 \le 0.7$. We see, therefore, that Gurevich uses a somewhat stronger temperature dependence for recombination than suggested by other authors, particularly for recombination with nitric oxide. The actual dependences are uncertain, but Gurevich's formula--which we use--predicts a density perturbation n that is on the high side of the uncertainty range.

IV. SOLUTIONS FOR SOME SPECIAL CASES

To illustrate the dependence of electron temperature and density on the various coefficients, solutions are presented for idealized cases. The equations of Sec. II can be integrated for these cases analvtically.

LIMITING CASE: NO HEAT CONDUCTION OR DIFFUSION

If heat conduction and diffusion are neglected, which is valid if the dimension of the heated region far exceeds the lengths L_N and L_T, then $L_T \approx L_N \approx 0$ and Eqs. (4) and (18) become

$$\frac{d\Delta T}{dt} = \frac{T_o}{\tau_T}\left[\frac{E^2}{E_p^2} - \frac{\Delta T}{T_o}\right] \tag{24}$$

and

$$\frac{dn}{dt} \approx -(2\alpha_o N_o + \beta)n - [\alpha(T) - \alpha(T_o)]N_o^2 \ . \tag{25}$$

The solution to Eq. (24),

$$\frac{\Delta T}{T_o} = \frac{E^2}{E_p^2}\left[1 - \exp(-t/\tau_T)\right] \ , \tag{26}$$

which, for late times, becomes

$$\frac{\Delta T}{T_o} \xrightarrow[t \gg \tau_T]{} \frac{E^2}{E_p^2} \ . \tag{27}$$

Thus, in the absence of diffusion or heat conduction, the relative temperature change is simply the squared ratio of the modifying field E to the characteristic field E_p. In that limit, the heated region is "frozen" and has spatial features identical to those of E^2.

In the same steady-state limit, the solution to Eq. (25) is

$$n \xrightarrow[t \gg \tau_N]{} \frac{N_o^2[\alpha(T_o) - \alpha(T)]}{2\alpha_o N_o + \beta} \quad . \tag{28}$$

The important result of Eq. (28) is that if there were no diffusion, the electron density perturbation n would occur solely through the temperature dependence of the recombination coefficient α. Because recombination is unimportant above about 200 km, virtually the entire electron density perturbation at such high altitudes depends upon ambipolar diffusion, omitted from Eq. (28). Stated differently, at altitudes much above 200 km there would be no perturbation if there were no diffusion.

SPATIALLY UNIFORM HEATING PULSE

If the heating pulse has a "square-wave" shape given by,

$$E(x) = \begin{cases} E_o & |x| \le a \\ 0 & |x| > a \ , \end{cases} \tag{29}$$

where E_o is constant and 2a is the width of the heated region, then Eqs. (7), (8), and (19) can be integrated straightforwardly.

It is convenient to normalize the fractional temperature and density changes to their values in the absence of transport processes and recombination. We therefore write

$$\Delta\tilde{T} = (\Delta T/T_o)/(E^2/E_p^2) \ , \tag{30}$$

and

$$\tilde{n} = (n/N_o)/(E^2/E_p^2) \ . \tag{31}$$

Without heat conduction, diffusion, or a temperature-dependent recombination rate, the normalized temperature and density perturbations are

$$
\begin{aligned}
\Delta\tilde{T}_e &= 1 , \\
\tilde{n} &= 0 \qquad |x| \le a , \\
\Delta\tilde{T}_e &= \tilde{n} = 0 \qquad |x| > a .
\end{aligned}
\tag{32}
$$

If we use the above form, Eq. (7), for the time-dependent temperature, becomes

$$
\Delta\tilde{T}_e = c_o + \frac{e^{-t/\tau_T}}{2} \left\{ s_o \operatorname{erf} \sqrt{\frac{b_+}{t}} + s_o \frac{e^{-b_+/t}}{2} \left[w(i\zeta_+) + w(i\zeta'_+) \right] \right.
$$

$$
\left. - \operatorname{erf} \sqrt{\frac{b_-}{t}} - \frac{e^{-b_-/t}}{2} \left[w(i\zeta_-) + w(i\zeta'_-) \right] \right\} , \tag{33}
$$

where w is the error function for complex arguments [15] and

$$
b_\pm = \tau_T \left(\frac{|x| \pm a}{2L_T} \right)^2 ,
$$

$$
\zeta_\pm = \sqrt{t/\tau_T} + \sqrt{b_\pm/t} ,
$$

$$
\zeta'_\pm = - \sqrt{t/\tau_t} + \sqrt{b_\pm/t} ,
$$

$$
c_o = \begin{cases} 1 & |x| \le a , \\ 0 & |x| > a , \end{cases}
$$

$$
s_o = \begin{cases} -1 & |x| \le a , \\ 1 & |x| > a . \end{cases}
$$

Equation (8), for the stationary perturbation temperature, becomes

$$\Delta\tilde{T}_e = G(x, L_T) \quad , \tag{34}$$

where

$$G\left(x, L_T\right) = c_o + \frac{1}{2} s_o \exp\left(-s_o \frac{|x| - a}{L_T}\right) - \exp\left(- \frac{|x| + a}{L_T}\right)$$

or

$$G(x, L_T) = \begin{cases} 1 - e^{-a/L_T} \cosh x/L_T & |x| \le a \; , \\ e^{-|x|/L_T} \sinh a/L_T & |x| > a \; . \end{cases}$$

Equation (19), for the stationary electron concentration, becomes

$$\tilde{n} = \gamma\, G\left(x, L_N\right) + \left(R_T\rho + \gamma \frac{G(x, L_T) - G(x, L_N)}{1 - \rho}\right) \; . \tag{35}$$

where

$$\rho = L_N^2/L_T^2 \; .$$

Equation (35) is undefined for $\rho = 1$. In this case, for $|x| \le a$,

$$\tilde{n} = \gamma G\left(x, L_N\right) + \left(R_T\rho + \gamma\right) \frac{e^{-a/L_N}}{2} \left(\frac{x}{L_N} \sinh \frac{x}{L_N} - \frac{a}{L_N} \cosh \frac{x}{L_N}\right) \tag{36}$$

and for $|x| > a$,

$$\bar{n} = \gamma G\left(x,\ L_N\right) + \left(R_T\rho + \gamma\right) \frac{e^{-|x|/L_N}}{2} \left(\frac{|x|}{L_N} \sinh \frac{a}{L_N} - \frac{a}{L_N} \cosh \frac{a}{L_N}\right) . \tag{37}$$

Peak Temperature and Density Perturbation in Limits of Weak and Strong Transport

The peak values of ΔT and n usually occur at the origin ($x = 0$). Setting $x = 0$ in Eqs. (34) and (35) and reverting to temperature and density that are not normalized, we find

$$\frac{\Delta T}{T_o} \xrightarrow[x\ =\ 0]{} \frac{E_o^2}{E_p^2} \left(1 - e^{-a/L_T}\right) \tag{38}$$

and

$$\frac{n}{N_o} \xrightarrow[x\ =\ 0]{} \frac{E_o^2}{E_p^2} \left[\gamma\left(1 - e^{-a/L_N}\right) + \frac{R_T L_N^2 + \gamma L_T^2}{L_T^2 - L_N^2} \left(e^{-a/L_N} - e^{-a/L_T}\right)\right] . \tag{39}$$

Equation (38) shows that heat conduction reduces the peak temperature by the factor $\left(1 - e^{-a/L_T}\right)$.

The limit of weak heat conduction and diffusion is reached when $a \gg L_N$ and L_T, which can occur either if the heating pulse is very wide (large a) or the diffusion and conduction lengths are small (small L). In that limit, Eqs. (38) and (39) become

$$\frac{\Delta T}{T_o} \xrightarrow[a\ \gg\ L]{} \frac{E_o^2}{E_p^2} , \tag{40}$$

and

$$\frac{n}{N_o} \xrightarrow[a\ \gg\ L]{} \gamma \frac{E_o^2}{E_p^2} . \tag{41}$$

which agree with Eqs. (27) and (28). From Table 2, we know that γ is negligible above 200 km.

Throughout much of the F-region, air-chemistry loss rates are so small and insensitive to temperature that γ can be assumed to equal zero, in which case Eq. (39) becomes

$$\frac{n}{N_o} = \left(\frac{R_T L_N^2}{L_T^2 - L_N^2}\right)\left(e^{-a/L_N} - e^{-a/L_T}\right) . \tag{42}$$

To examine the limit of strong transport, we assume L_N and $L_T >> a$, in which case Eqs. (38) and (39) become

$$\frac{\Delta T}{T_o} \approx \left(\frac{a}{L_T}\right)\left(\frac{E_o^2}{E_p^2}\right) , \tag{43}$$

$$\left|\frac{n}{N_o}\right| \approx \frac{aR_T}{L_T}\left(\frac{L_N}{L_T + L_N}\right)\frac{E_o^2}{E_p^2} \tag{44}$$

$$= R_T\left(\frac{L_N}{L_T + L_N}\right)\frac{\Delta T}{T_o} , \tag{45}$$

where, from Table 2, $L_N \leq L_T$. Equations (44) and (45) show that strong heat conduction reduces the peak temperature perturbation by the factor a/L_T. Moreover, n/N_o is small whether transport processes are weak or strong, because:

(*a*) If transport processes are weak, then, according to Eq. (41), $(n/N_o)/(\Delta T/T_o) = \gamma << 1$.

(*b*) If diffusion is strong, then $n/N_o \lesssim \Delta T/T_o$, per Eq. (45). But, according to Eq. (43), $\Delta T/T_o$ itself is then of order a/L_T, which is small.

Transport, which is necessary to create a density perturbation in much of the F-layer, therefore suppresses the temperature increase ΔT that drives the density perturbation. Although large values of ΔT can be created if the heating width 2a is great enough, the fractional density change n/N_0 is always much smaller than unity.

Numerical Examples

To illustrate the temperature behavior, we choose $L_T = 5$ km, a somewhat lower value of L_T than occurs for the F-region. Figure 1 shows the temperature perturbation caused by uniform heating for 10 s in a region 40 km wide (a = 20 km). The thermal response time is $\tau_T =$ 5 s. After 2 s, the interior of the heated region has a nearly uniform temperature, because it is wider than the conductivity length L_T, and has not been heated long enough to produce gradients large enough for heat conductivity to alter the temperature appreciably. Later, as temperature gradients increase, the high temperatures at the edges of the heated region are smoothed by conduction. The total heating time is about twice the thermal response time, and the central temperature attains 85 percent of its stationary value after 10 s. The maximum stationary value of the temperature perturbation is about the same as would be achieved without conductivity, because the conductivity length is smaller than the width of the heated region.

Figure 2 illustrates the effect of beam width on electron temperature. In this case, $L_T = 17$ km, which is the actual conductivity length for 200-km altitude during the day (Table 2). Figures 2a-c, show the temperature change resulting from uniformly heated regimes that are 20-, 5-, and 1-km wide. The final temperature perturbation is much smaller than that attained in the nonconductive limit because even the widest beam is comparable in dimension to the conductivity length. Thus, for all cases shown, the heated region is too narrow relative to L_T to sustain the full temperature perturbation; the narrower the heater beams, the more the temperature perturbation decreases. For the smallest (1 km) heating width, the peak temperature perturbation is less than 5 percent of that in the nonconducting limit. Table 2 shows that L_T during the day at 200-km altitude is

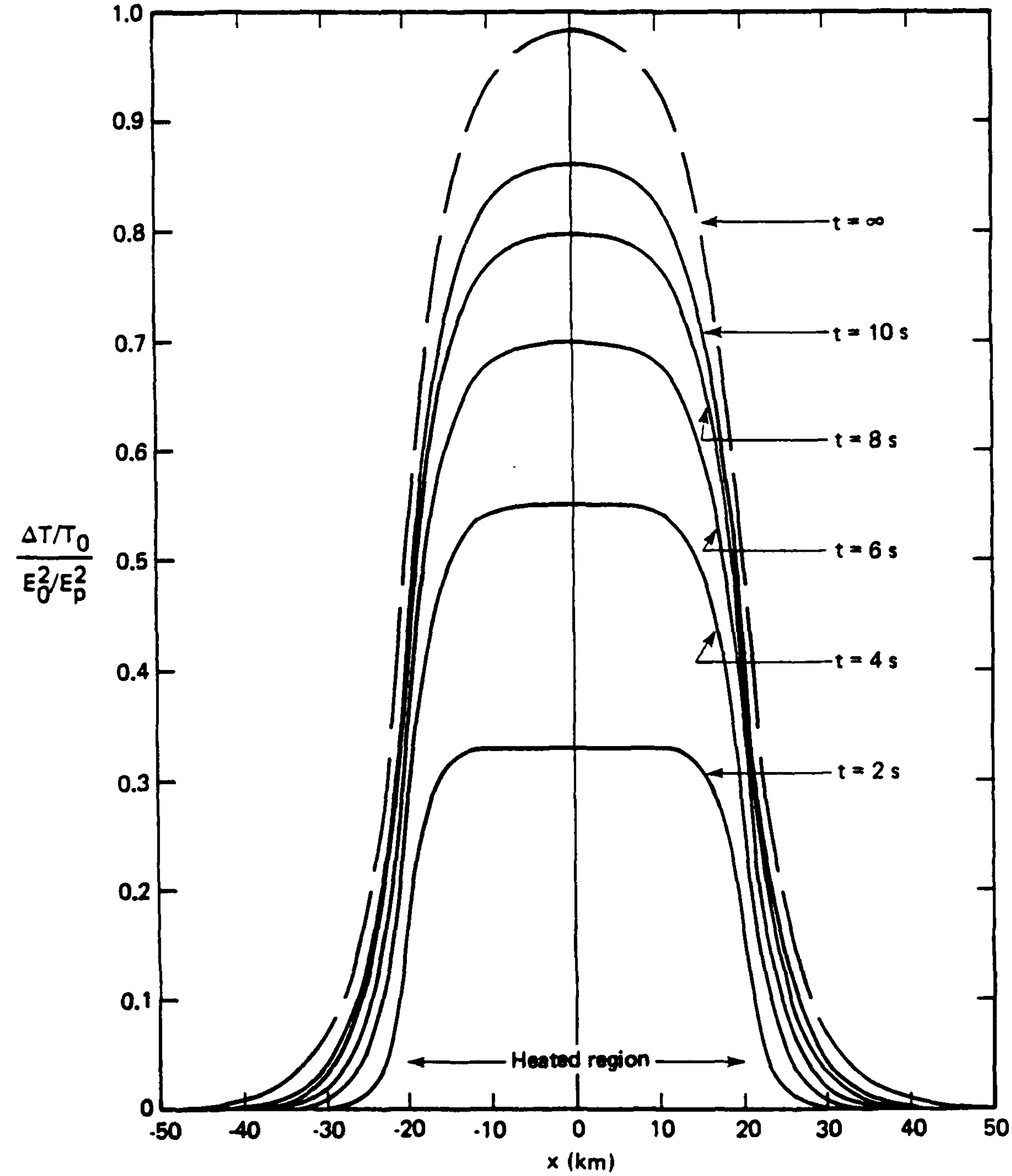

Fig. 1. Electron temperature change: $a = 20$ km, $L_T = 5$ km, $\tau_T = 5$ s.

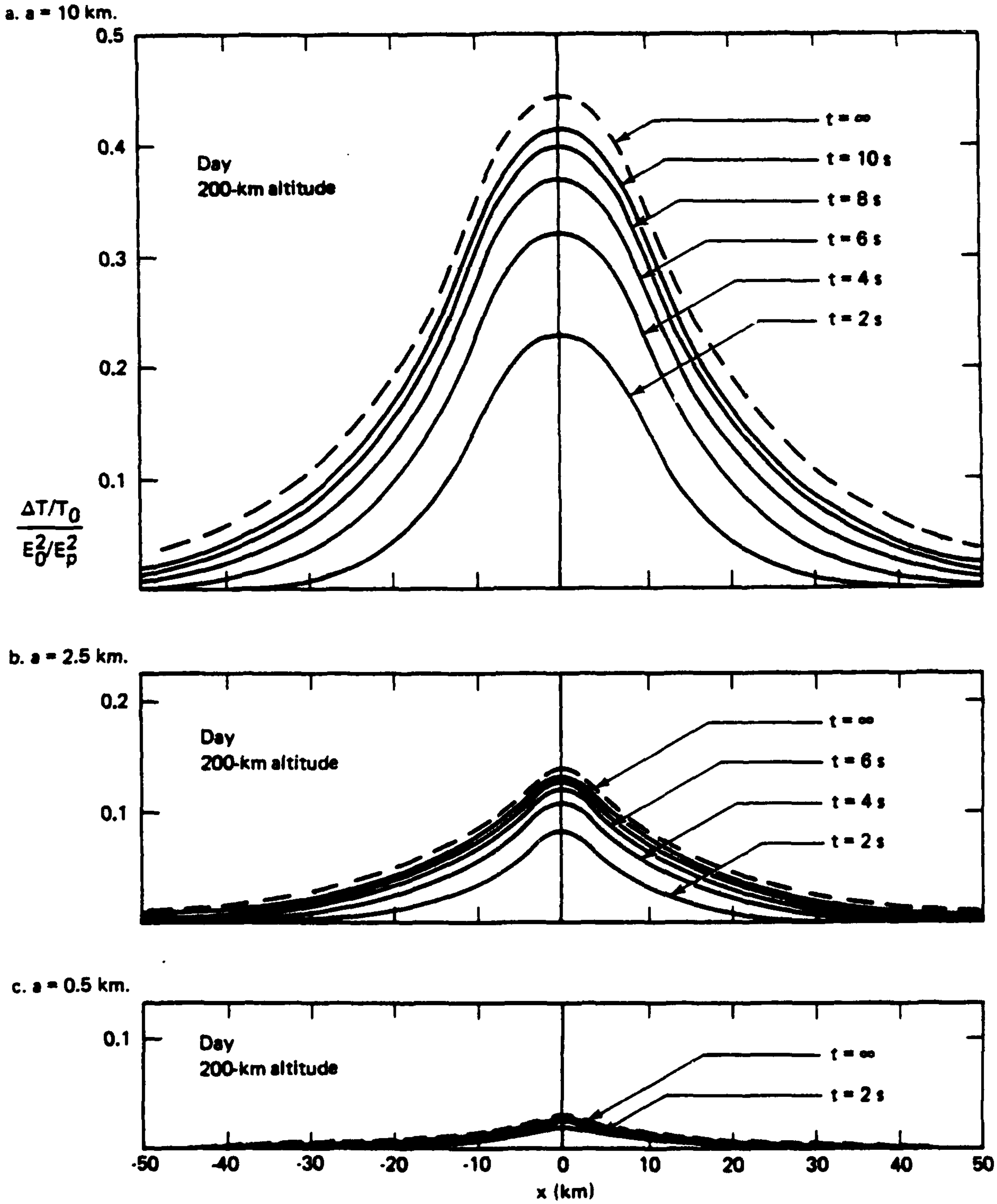

Fig. 2. Electron temperature change for three beam widths: $\tau_T = 5.2$ s $> L_T = 17$ km.

less than that at any other F-region altitude, day or night. The normalized temperature perturbation therefore would be even smaller at greater altitudes or at night, than at 200 km during the day.

Figure 3 shows the electron temperature perturbation at low, medium, and high F-region altitudes for a 20-km wide heating beam during the day. Parameter values are given in Table 2. As previously stated, the perturbation is greatest at 200 km. As the conductivity length increases with altitude, the amplitude of the temperature perturbation decreases. Table 2 shows that the thermal response time increases with altitude, because collision frequency decreases with altitude. It thus takes the temperature longer to reach its stationary value as altitude increases.

The main question is whether a heater can produce electron density gradients that are large enough to focus or defocus an incident beam and alter the received sky wave at a ground-based receiver. Figure 4 shows the daytime stationary temperature and density perturbations at 200-km altitude for three values of width 2a. At 200-km, an increase in temperature causes a decrease in the recombination rate which, in turn, causes an increase in electron density n.

Two diffusion processes determine the electron density change: (1) thermal diffusion, where electrons of high thermal velocity leave the heated region (equivalent to thermal expansion); and (2) ambipolar diffusion, where electrons in regions of high density diffuse to regions of lower concentration. The two processes oppose each other --that is one reason why density changes are small.

Figures 5 and 6 show the stationary daytime temperature and density changes at 250 km and 300 km, respectively, during the day for 1-, 5-, and 20-km heated widths. The recombination rate is little affected by temperature changes, so $\gamma \approx o$ above 200 km and high temperatures produce a density reduction due to thermal diffusion. Accompanying the concentration deficit within the heating region is a concentration excess in the surrounding region. The excess concentration is required for overall conservation of electrons because, in the absence of ion-pair production or recombination changes, there can be no net change in the number of electrons.

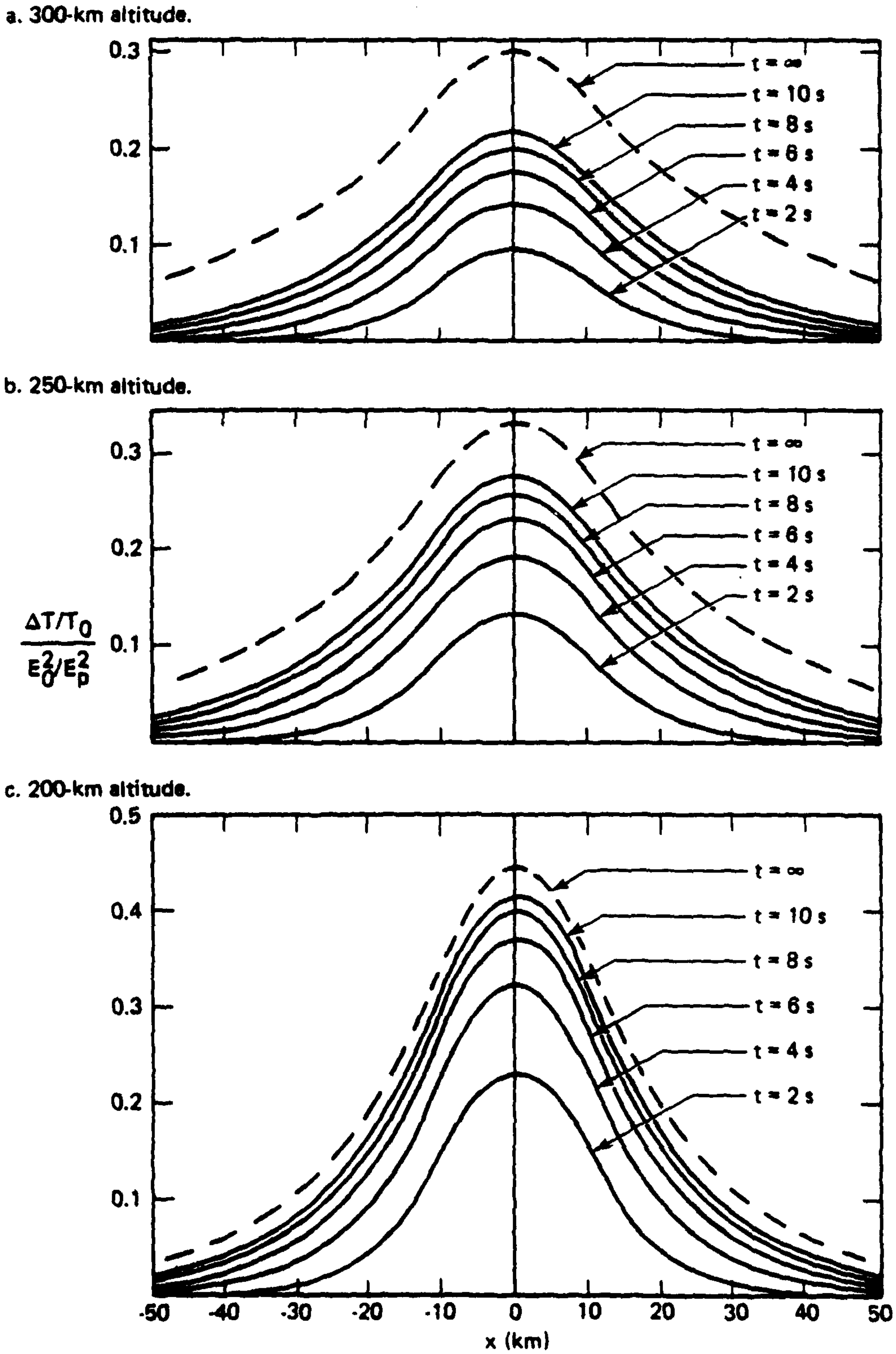

Fig. 3. Daytime electron temperature change at altitudes; a = 10 km.

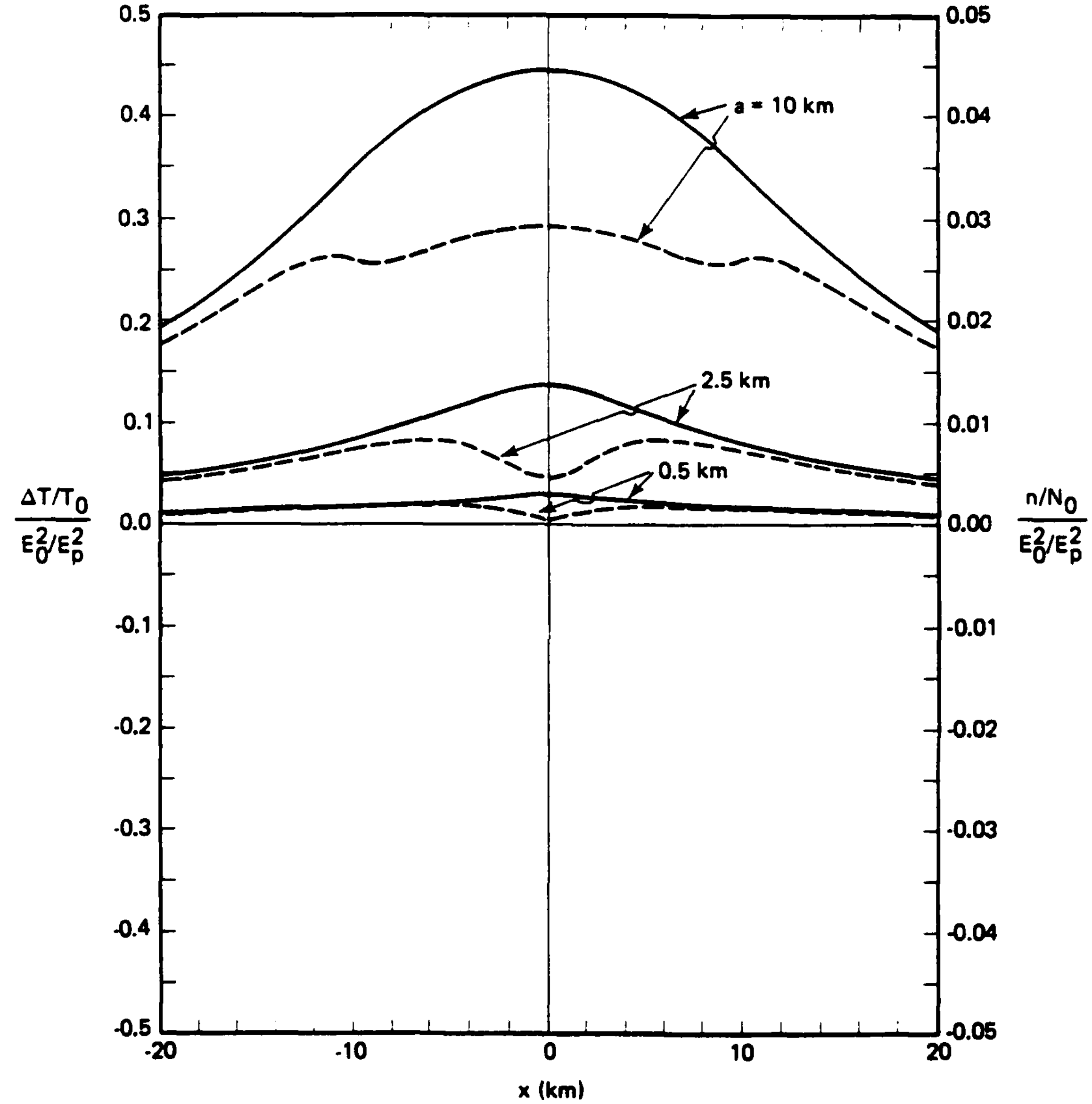

Fig. 4. Stationary temperature (solid) and electron density (dashed) changes at 200-km altitude during day for three heated widths and $\gamma = 0.08$.

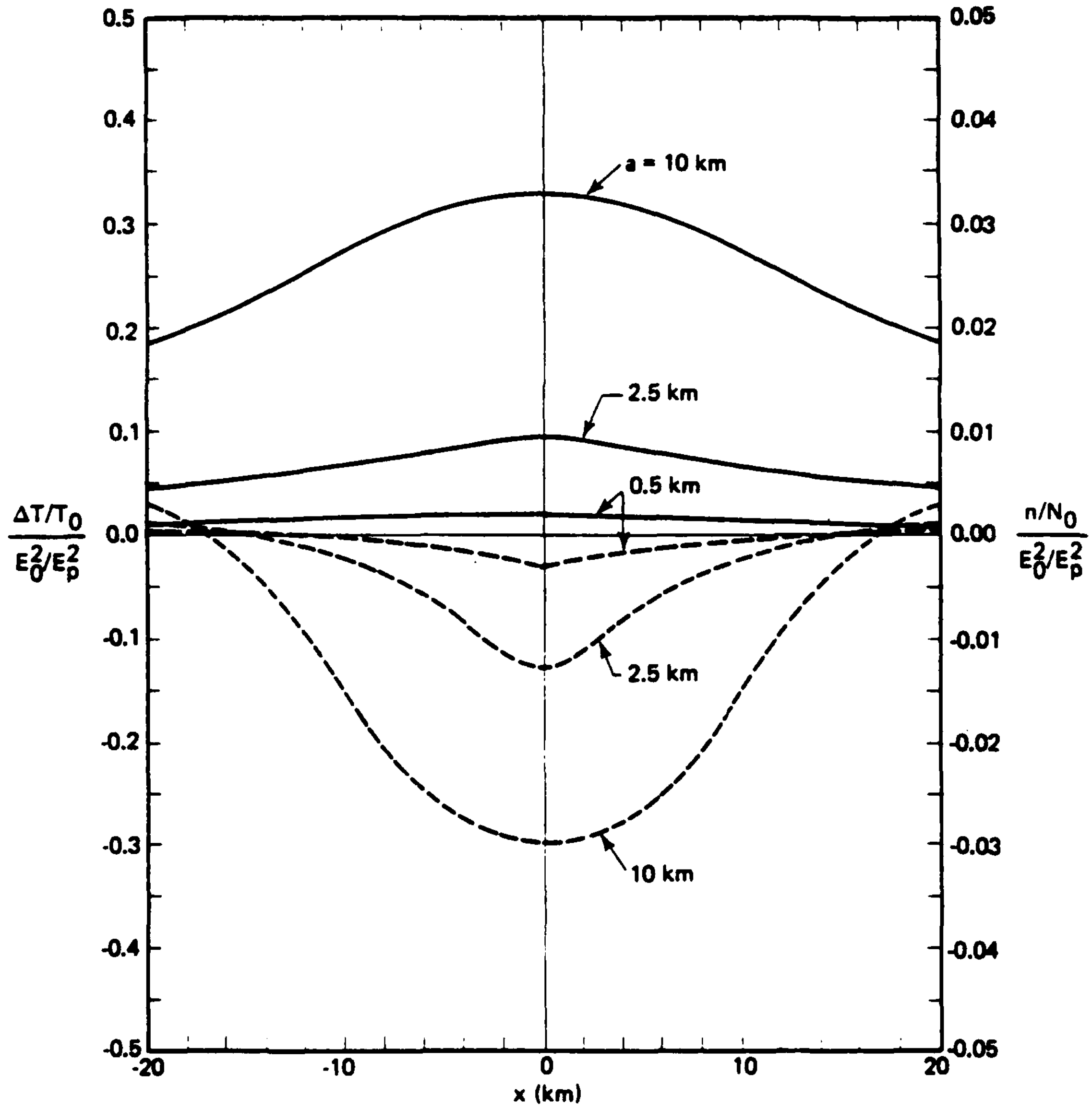

Fig. 5. Stationary temperature (solid) and concentration (dashed) changes at 250-km altitude during day for three heated widths and $\gamma = 0$.

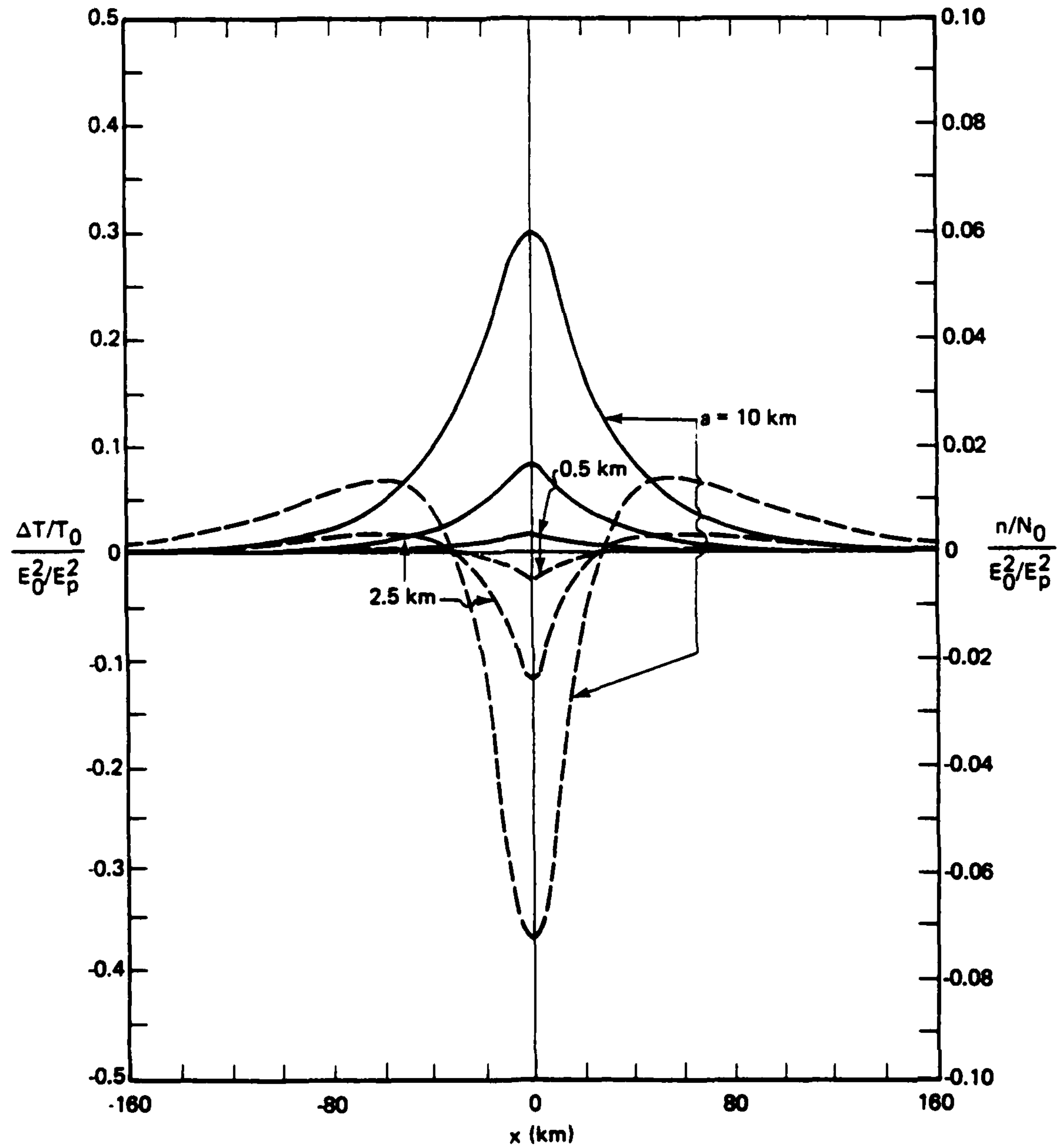

Fig. 6. Stationary temperature (solid) and electron (dashed) changes at 300-km altitude during day for three heated widths and $\gamma = 0$.

V. DISCUSSION

The transport and chemical equations that describe electron density and temperature changes produced by electron fields in the ionosphere have been derived and cast into a form suitable for numerical integration. By using those equations in conjunction with previously developed methods for calculating electric fields near caustics, the density and temperature changes produced by an oblique ionospheric HF heater can be computed straightforwardly. The necessary inputs are: (1) heater power, (2) heater radiation pattern, (3) heater frequency, and (4) model ionosphere.

Analytic solutions for idealized heater beam-shapes indicate that heat conduction and diffusion are the dominant processes above 220 km. Those processes suppress the temperature substantially, because the caustics are narrower than conduction lengths, but are necessary to produce density changes, which influence HF propagation. The fractional density changes are always small. At altitudes of 220 km and below, the dominant process is the heating-induced reduction of the electron-ion recombination rate which causes an increase in the electron density. For that reason, greater density changes can be produced at 200 km than at higher altitudes.

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Appendix A

SOLUTION TO EQ. (4)

The equation

$$\tau_T \frac{\partial \Delta T}{\partial t} - L_T^2 \frac{\partial^2 \Delta T}{\partial x^2} = T_o\left(\frac{E^2}{E_p^2}\right) ,$$

which describes the temperature perturbation of a plasma due to the electric field E, is of the general form:

equation: $u_t - ku_{xx} + \gamma u = Q(x,t)$,

boundary conditions: $u(x,o) = f(x)$ and $u \to o$ as $x \to \pm\infty$,

where $u_t \equiv \dfrac{\partial u}{\partial t}$,

$$u_{xx} \equiv \frac{\partial^2 u}{\partial x^2} .$$

We simplify the equation by multiplying it and its boundary condition by $e^{\gamma t}$, and let $u = e^{-\gamma t}w(x,t)$. Then,

$$\mu_t - ku_{xx} + \gamma u = e^{-\gamma t} (w_t - \gamma w - kw_{xx} + \gamma w) = Q(x,t) ,$$

so,

$$w_t - kw_{xx} = e^{\gamma t} Q(x,t) = R(x,t) \qquad \text{(A.1)}$$

and,

$$w(x,o) = u(x,o) = f(x).$$

We claim that the solution to Eq. (A.1) is as follows:

$$w(x,t) = \int_0^t ds \int_{-\infty}^{\infty} d\xi \, \frac{e^{-[(x-\xi)^2/4k(t-s)]}}{\sqrt{4\pi k(t-s)}} R(\xi,s) + \int_{-\infty}^{\infty} d\xi \, \frac{e^{-[(x-\xi)^2/4kt]}}{\sqrt{4\pi kt}} f(\xi) \; . \tag{A.2}$$

where $R(\xi,s) = e^{\gamma s} Q(\xi,s)$;

hence,

$$u(x,t) = e^{-\gamma t} w(x,t)$$

$$= \int_0^t ds \int_{-\infty}^{\infty} d\xi \, \frac{\exp\left[-\gamma(t-s) - \dfrac{(x-\xi)^2}{4k(t-s)}\right]}{\sqrt{4\pi k(t-s)}} Q(\xi,s)$$

$$+ e^{-\gamma t} \int_{-\infty}^{\infty} d\xi \, \frac{e^{-[(x-\xi)^2/4kt]}}{\sqrt{4\pi kt}} f(\xi) \; .$$

PROOF OF EQ. (A.2)

Let

$$w_t - kw_{xx} = R(x,t) \; ,$$

$$w(x,o) = f(x) \; ,$$

$$\frac{\partial w}{\partial t} \equiv w_t \; ,$$

$$\frac{\partial^2 w}{\partial x^2} \equiv w_{xx} \; .$$

We assume that w, R, f, and their second derivatives are square-integrable on $(-\infty, \infty)$. Let

$$\hat{w}(\lambda,t) = \int_{-\infty}^{\infty} e^{-i\lambda x} w(x,t) \, dx$$

denote the Fourier transform of $w(x,t)$. Then,

$$\hat{w}_t(\lambda,t) = \int_{-\infty}^{\infty} e^{-i\lambda x}\, w_t(x,t)\, dx = \int_{-\infty}^{\infty} e^{-i\lambda x} \left[k w_{xx} + R(x,t)\right] dx$$

$$= k \cdot \int_{-\infty}^{\infty} w_{xx}\, e^{-i\lambda x}\, dx + \int_{-\infty}^{\infty} e^{-i\lambda x}\, R(x,t)\, dx \; .$$

Assuming $\lim_{|x|\to\infty} w(x,t) = 0$, and $\lim_{|x|\to\infty} w_x(x,t) = 0$, the first term can be integrated twice by parts. The second term is the Fourier transform of $R(x,t)$, i.e., $\hat{R}(\lambda,t)$. We get the following:

$$\frac{d}{dt}\hat{w}(\lambda,t) = -k\lambda^2 \hat{w}(\lambda,t) + \hat{R}(\lambda,t) \; ,$$

$$\hat{w}(\lambda,0) = \hat{f}(\lambda) \; ,$$

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x}\, f(x)\, dx \; .$$

This is an ordinary differential equation for $\hat{w}(\lambda,t)$. Its solution is

$$\hat{w}(\lambda,t) = e^{-\lambda^2 kt} \int_0^t e^{\lambda^2 ks}\, \hat{R}(\lambda,s)\, ds + e^{-\lambda^2 kt}\, \hat{f}(\lambda) \; .$$

Consequently, inverting the Fourier transform, we have:

$$w(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x} \left[e^{-\lambda^2 kt} \int_0^t e^{\lambda^2 ks} \hat{R}(\lambda,s)\, ds + e^{-\lambda^2 kt} \dot{f}(\lambda) \right] d\lambda \ .$$

$$w(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x} e^{-\lambda^2 kt} \hat{f}(\lambda)\, d\lambda + \frac{1}{2\pi} \int_{\infty}^{\infty} e^{i\lambda x - \lambda^2 kt}$$

$$\cdot \int_0^t e^{\lambda^2 ks} \hat{R}(\lambda,s)\, ds\, d\lambda \ .$$

The convolution theorem for Fourier transforms gives:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x} \hat{g}(\lambda;t)\, \hat{h}(\lambda;t)\, d\lambda = g * h(x,t)$$

$$= \int_{-\infty}^{\infty} g(\xi,t)\, h(x-\xi,t)\, d\xi \ ,$$

and therefore,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x} \left[e^{-\lambda^2 kt} \hat{f}(\lambda) \right] d\lambda = \int_{-\infty}^{\infty} d\xi\, f(\xi)\, \frac{e^{-[(x-\xi)^2/4kt]}}{\sqrt{4\pi kt}} \ .$$

The second integral,

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x} e^{-\lambda^2 kt} \int_0^t e^{\lambda^2 ks} \hat{R}(\lambda, s)\, ds\, d\lambda ,$$

can be written (assuming uniform convergence) as follows:

$$\frac{1}{2\pi} \int_0^t ds \int_{-\infty}^{\infty} e^{-i\lambda x} e^{-[\lambda^2 k(t-s)]} \hat{R}(\lambda, s)\, d\lambda$$

which, by the convolution theorem, is:

$$\int_0^t ds \int_{-\infty}^{\infty} \frac{e^{-[(x-\xi)^2/4k(t-s]}}{\sqrt{4\pi k(t-s)}} R(\xi, s)\, d\xi .$$

In terms of the electron temperature problem,

$$u_t - ku_{xx} + \gamma u = Q(x,t) ,$$

$$u(x,o) = f(x) ,$$

corresponds to

$$\frac{\partial}{\partial t}(T - T_{eo}) - \frac{L_T^2}{\tau_T} \frac{\partial^2}{\partial x^2}(T - T_{eo}) = - \frac{(T - T_{eo})}{\tau_T} + \frac{T_{eo}}{\tau_T} \frac{E^2(x,t)}{E_p^2} ,$$

$$T(x,o) = T_{eo} ,$$

or,

$$\frac{\partial T}{\partial t} - \frac{L_T^2}{\tau_T} \frac{\partial^2 T}{\partial x^2} + \frac{1}{\tau_T} T = \frac{T_{eo}}{\tau_T} \left[1 + \frac{E^2(x,t)}{E_p^2} \right] ,$$

$$T(x,o) = T_{eo} ,$$

the solution of which is, by Eq. (A.2):

$$T(x,t) = e^{-(t/\tau_T)} \left\{ \int_0^t ds \int_{-\infty}^{\infty} \frac{e^{-[(x-\xi)^2/4k(t-s)]}}{\sqrt{4\pi\kappa(t-s)}} e^{\frac{s}{\tau_T}} \cdot \frac{T_{eo}}{\tau_T} \left[1 + \frac{E^2(\xi,s)}{E_p^2} \right] d\xi \right. .$$

$$\left. + \int_{-\infty}^{\infty} d\xi \; \frac{e^{-[(x-\xi)^2/4kt]}}{\sqrt{4\pi\kappa t}} T_{eo} \right\} , \qquad \text{(A.3)}$$

where $\kappa = \frac{L_T^2}{\tau_T}$.

Equation (A.3), by virtue of the identity,

$$\int_{-\infty}^{\infty} dz \; e^{-[(x-z)^2/u^2]} = \sqrt{\pi}\, u ,$$

is equivalent to

$$T(x,t) = T_{eo} \left\{ 1 + \frac{1}{\tau_T} \int_0^t ds \int_{-\infty}^{\infty} \frac{\exp\left[- \frac{(t-s)}{\tau_T} - \frac{(x-\xi)^2}{4\kappa(t-s)} \right]}{\sqrt{4\pi\kappa(t-s)}} \frac{E^2(\xi,s)}{E_p^2} d\xi \right\} .$$

Appendix B

ASYMPTOTIC LIMIT TO EQ. (7)

We analyze the asymptotic behavior of the expression

$$T(x,t) = T_{eo}\left\{1 + \frac{1}{\tau_T}\int_0^t ds \int_{-\infty}^{\infty} \frac{\exp\left[-\dfrac{(t-s)}{\tau_T} - \dfrac{(x-\xi)^2}{4\kappa(t-s)}\right]}{\sqrt{4\pi\kappa(t-s)}} Q(\xi)\, d\xi\right\} . \tag{B.1}$$

For t large, where $\kappa = L_T^2/\tau_T$,

$$Q(\xi) = \frac{E^2(\xi)}{E_p^2} .$$

We also consider the following:

$$\lim_{t\to\infty} \frac{1}{\tau_T}\int_0^t ds \int_{-\infty}^{\infty} \frac{\exp\left[-\dfrac{(t-s)}{\tau_T} - \dfrac{(x-\xi)^2}{4\kappa(t-s)}\right]}{\sqrt{4\pi\kappa(t-s)}} Q(\xi)\, d\xi . \tag{B.2}$$

Reversing the order of integration gives:

$$\frac{1}{\tau_T}\int_{-\infty}^{\infty} Q(\xi)\left\{\lim_{t\to\infty}\int_0^t \frac{\exp\left[-\dfrac{(t-s)}{\tau_T} - \dfrac{(x-\xi)^2}{4\kappa(t-s)}\right]}{\sqrt{4\pi\kappa(t-s)}} ds\right\} d\xi$$

$$= \frac{1}{\sqrt{\pi}\ \tau_T} \int_{\infty}^{\infty} Q(\xi) \left\{ \lim_{u \to \infty} \int_0^u \frac{\exp\left[- \frac{u}{\tau_T} - \frac{(x - \xi)^2}{4\kappa u}\right]}{\sqrt{4\kappa u}} du \right\} d\xi \ , \qquad (B.3)$$

now

$$\int_0^u \frac{\exp\left[- \frac{u}{\tau_T} - \frac{(x - \xi)^2}{4\kappa u}\right]}{\sqrt{4\kappa u}} du = \int_0^u \exp\left[- \frac{4\kappa u}{4\kappa \tau_T} - \frac{(x-\xi)^2}{4\kappa u}\right] du \ . \qquad (B.4)$$

We let $u' = 4\kappa u$ in Eq. (B.4) and obtain

$$\frac{1}{4\kappa} \int_0^u \frac{\exp\left[- \frac{u'}{4\kappa \tau_T} - \frac{(x-\xi)^2}{u'}\right]}{\sqrt{u'}} du' . \qquad (B.5)$$

Now set $u' = w^2$, $du' = 2w\ du = 2\sqrt{u'}\ dw$. Then Eq. (B.5) becomes:

$$\frac{1}{4\kappa} \cdot 2 \int_0^{\sqrt{4\kappa u}} \exp\left[- \frac{w^2}{4\kappa \tau_T} - \frac{(x-\xi)^2}{w^2}\right] dw \ . \qquad (B.6)$$

From Abramowitz and Stegun [15], we know that:

$$\int e^{[-a^2 w^2 - (b^2/w^2)]}\ dw = \frac{\sqrt{a}}{4\pi} \left[e^{2ab} \operatorname{erf}\left(aw + \frac{b}{w}\right) + e^{-2ab} \operatorname{erf}\left(aw - \frac{b}{w}\right)\right] + C \ ,$$

where

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2} \, ds \quad ,$$

and,

$$a = \frac{1}{\sqrt{4\kappa\tau_T}} \quad ,$$

$$b = \sqrt{(x-\xi)^2} = |(x-\xi)| \quad .$$

Hence, from Eq. (B.6) we get the following:

$$\int_0^{\sqrt{4\kappa u}} \exp\left[-\frac{w^2}{4\kappa\tau_T} - \frac{(x-\xi)^2}{w^2}\right] dw$$

$$= \frac{\sqrt{4\pi\kappa\tau}_T}{4} \left\{ e^{|x-\xi|/\sqrt{\kappa\tau}_T} \operatorname{erf}\left(\frac{u}{\sqrt{4\kappa\tau_T}} + \frac{|x-\xi|}{w}\right) + e^{|x-\xi|/\sqrt{\kappa\tau}_T}\right.$$

$$\left.\left. \cdot \operatorname{erf}\left(\frac{w}{\sqrt{4\kappa\tau_T}} - \frac{|x-\xi|}{w}\right)\right\} \right]_0^{\sqrt{4\kappa u}}$$

$$= \frac{\sqrt{\pi\kappa\tau}_T}{4} \left[e^{|x-\xi|/\sqrt{\kappa\tau}_T} \operatorname{erf}\left(\frac{u}{\sqrt{\tau}_T} + \frac{|x-\xi|}{\sqrt{4\kappa u}}\right) + e^{|x-\xi|/\sqrt{\kappa\tau}_T}\right.$$

$$\left. \cdot \operatorname{erf}\left(\frac{u}{\sqrt{\tau}_T} - \frac{|x-\xi|}{\sqrt{4\kappa u}}\right) - e^{|x-\xi|/\sqrt{\kappa\tau}_T} \operatorname{erf}(+\infty) - e^{|x-\xi|/\sqrt{\kappa\tau}_T} \operatorname{erf}(-\infty)\right]$$

$$= \frac{\sqrt{4\pi\kappa\tau_T}}{4}\left\{ e^{|x-\xi|/\sqrt{\kappa\tau_T}}\left[1 + \mathrm{erf}\left(\frac{u}{\sqrt{\tau_T}} - \frac{|x-\xi|}{\sqrt{4\kappa u}}\right)\right] - e^{|x-\xi|/\sqrt{\kappa\tau_T}}\right.$$

$$\left. \cdot\ \mathrm{erfc}\left(\frac{u}{\sqrt{\tau_T}} - \frac{|x-\xi|}{\sqrt{4\kappa u}}\right)\right\} , \tag{B.7}$$

where $\mathrm{erfc}(u) = 1 - \mathrm{erf}(u)$.

As we take $\lim_{t\to\infty}$, $u \to \infty$ and the above expression tends toward

$$\frac{\sqrt{4\pi\kappa\tau_T}}{4} \cdot 2\ e^{-(|x-\xi|/\sqrt{\kappa\tau_T})} . \tag{B.8}$$

Hence the expression we seek is:

$$\lim_{u\to\infty} \int_0^u \frac{\exp\left[-\frac{u}{\tau_T} - \frac{(x-\xi)^2}{4\kappa u}\right]}{\sqrt{4\kappa u}}\, dw$$

$$= \frac{2}{4\kappa} \lim_{u\to\infty} \int_0^{\sqrt{4\kappa u}} \exp\left[-\frac{w^2}{4\kappa\tau_T} - \frac{(x-\xi)^2}{w^2}\right] dw$$

$$= \frac{2}{4\kappa} \cdot \frac{\sqrt{4\pi\kappa\tau_T}}{4} \cdot 2\ e^{-(|x-\xi|/\sqrt{\kappa\tau_T})}$$

$$= \frac{\sqrt{4\pi\kappa\tau_T}}{4\kappa}\ e^{-(|x-\xi|/\sqrt{\kappa\tau_T})} . \tag{B.9}$$

We substitute Eq. (B.9) back into Eq. (B.3) to get:

$$\frac{\sqrt{4\pi\kappa\tau_T}}{4\sqrt{\pi}\ \tau_T\kappa} \int_{-\infty}^{\infty} Q(\xi)\ e^{-(|x-\xi|/\sqrt{\kappa\tau_T})}\ d\xi$$

$$= \frac{1}{2\sqrt{\kappa\tau_T}} \int_{-\infty}^{\infty} Q(\xi)\ e^{-(|x-\xi|/\sqrt{\kappa\tau_T})}\ d\xi\ .$$

Hence, we find that

$$\lim_{t\to\infty} T(x,t) = T_{co}\left[1 + \frac{1}{2\sqrt{\kappa\tau_T}} \int_{-\infty}^{\infty} Q(\xi)\ e^{-(|x-\xi|/\sqrt{\kappa\tau_T})}\ d\xi\right] ,$$

and

$$\kappa = \frac{L_T2}{\tau_T}\ .$$

It is then clear, as we consider the limiting behavior as $\kappa \to 0$ (i.e., no diffusion), that

$$\lim_{\substack{t\to\infty \\ \kappa\to 0}} T(x,t) = T_{eo}\left[1 + \frac{E^2(x)}{Ep^2}\right] ,$$

since the kernel

$$\frac{e^{-(|x-\xi|/\sqrt{\kappa\tau_T})}}{2\sqrt{\kappa\tau_T}}$$

becomes a δ-function as $\kappa \to 0$.

Appendix C

STEADY-STATE SOLUTION TO EQ. (18)

The steady-state form of Eq. (18) is:

$$L_N^2 \frac{d^2n}{dx^2} + R_T L_N^2 \frac{N_0}{T_{eo}} \frac{d^2\Delta T_e}{dx^2} - n = -\gamma \frac{N_0}{T_{eo}} \Delta T_e \ .$$

for boundary conditions $\lim_{x\to\pm\infty} n = 0$. This is of the form

$$y'' - a^2 y = -f(x) \qquad y\ (\pm\infty) = 0 \ ,$$

and

$$a = \frac{1}{L_N} \ .$$

The solution is found by the method of variation of parameters by assuming $y = \mu_1 e^{ax} + \mu_2 e^{-ax}$. We solve the system thus:

$$y_1 = e^{ax} \ ,$$

$$y_2 = e^{-ax} \ ,$$

$$\mu_1' \ y_1 + \mu_2' \ y_2 = 0 \ ,$$

$$\mu_1' \ y_1' + u_2' \ y_2' = -f(x) \ ,$$

and find

$$\mu_2' = \frac{1}{2a} \ e^{ax} f(x) \qquad \mu_1' = -\frac{1}{2a} \ e^{-ax} f(x) \ .$$

whence

$$y = -\frac{e^{ax}}{2a}\int_{\xi_1}^{x} e^{-as}\ f(s)\ ds + \frac{e^{-ax}}{2a}\int_{\xi_2}^{x} e^{as}\ f(s)\ ds.$$

Putting $\xi_1 = +\infty$ and $\xi_2 = -\infty$ allows $\lim\limits_{x\to\pm\infty} y = 0$, and hence,

$$y = \frac{1}{2a}\int_{x}^{\infty} e^{-a(x-s)} f(s)\ ds + \frac{1}{2a}\int_{\infty}^{x} e^{-a(s-x)} f(s)\ ds$$

$$= \frac{1}{2a}\int_{\infty}^{\infty} e^{-a|x-s|}\ f(s)\ ds\ .$$

Now,

$$-f(x) = \left(R_T L_N^2 \frac{N_0}{T_{eo}} \frac{d^2 \Delta T_e}{dx^2} + \gamma \frac{N_0}{T_{eo}} \Delta T_e\right) \cdot \frac{1}{L_N^2}\ .$$

Using

$$\frac{d^2 \Delta T_e}{dx^2} = \frac{1}{L_T^2}\left(\Delta T_e - T_{eo} Q\right),$$

where $Q = E^2/E_p^2$,

we have:

$$-f(x) = \left[\frac{N_0}{T_{eo}}\left(\gamma + R_T \frac{L_N^2}{L_T^2}\right)\Delta T_e - R_T N_0 \frac{Q L_N^2}{L_T^2}\right]\frac{1}{L_N^2}\ .$$

We calculate

$$\frac{n}{N_0} = \frac{1}{2L_N} \int_{-\infty}^{\infty} \Psi(x')\, e^{-\frac{|x-x'|}{L_N}}\, dx' \ ,$$

where $$\Psi(x') = \frac{1}{T_{eo}} \left(\gamma + R_T \frac{L_N^2}{L_T^2} \right) \Delta T_e - R_T \frac{L_N^2}{L_T^2}\, Q \ ,$$

and $$\frac{\Delta T_e}{T_{eo}}(x') = \frac{1}{2L_T} \int_{-\infty}^{\infty} Q(y)\, e^{-\frac{|x'-y|}{L_T}}\, dy$$

(i.e., the steady-state temperature solution from Appendix B). Expanding, we have

$$\Psi(x') = \left(\gamma + R_T \frac{L_N^2}{L_T^2}\ \frac{1}{2L_T} \right) \int_{-\infty}^{\infty} Q(y)\, e^{-\frac{|x'-y|}{L_T}}\, dy - R_T \frac{L_N^2}{L_T^2}\ Q(x')$$

$$= \frac{\gamma}{2L_T} \int_{-\infty}^{\infty} Q(y)\, e^{-\frac{|x'-y|}{L_T}}\, dy + \frac{R_T L_N^2}{2L_T^3} \int_{-\infty}^{\infty} Q(y)\, e^{-\frac{|x'-y|}{L_T}}\, dy - R_T \frac{L_N^2}{L_T^2}\, Q(x')$$

$$= \frac{\gamma}{2L_T} \int_{-\infty}^{\infty} Q(y)\, e^{-\frac{|x'-y|}{L_T}}\, dy + \frac{R_T L_N^2}{L_T^2} \left[\frac{1}{2L_T} \int_{-\infty}^{\infty} Q(y)\, e^{-\frac{|x'-y|}{L_T}}\, dy - Q(x') \right] ,$$

then

$$
\frac{n}{N_0} = \frac{1}{2L_N} \int_{-\infty}^{\infty} \left\{ \frac{\gamma}{2L_T} \int_{-\infty}^{\infty} Q(y)\, e^{-\frac{|x'-y|}{L_T}}\, dy \right.
$$

$$
\left. + \frac{R_T L_N}{L_T^2} \left[\frac{1}{2L_T} \int_{-\infty}^{\infty} Q(y)\, e^{-\frac{|x'-y|}{L_T}}\, dy - Q(x') \right] \right\} e^{-\frac{|x-x'|}{L_N}}\, dx' \; ,
$$

and

$$
\frac{n}{N_0} = \frac{\gamma}{4L_N L_T} \cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q(y)\, e^{-\frac{|x'-y|}{L_T} - \frac{|x-x'|}{L_N}}\, dy\, dx'
$$

$$
+ \frac{R_T L_N}{4L_T^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q(y)\, e^{-\frac{|x'-y|}{L_T} - \frac{|x-x'|}{L_N}}\, dy\, dx'
$$

$$
- \frac{R_T L_N}{2L_T^2} \int_{-\infty}^{\infty} Q(x')\, e^{-\frac{|x-x'|}{L_N}}\, dx' \; .
$$

A lengthy but elementary calculation shows that

$$
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q(y)\, e^{-\frac{|x'-y|}{L_T} - \frac{|x-x'|}{L_N}}\, dx'\, dy
$$

$$
= \frac{2L_N L_T}{L_N^2 - L_T^2} \int_{-\infty}^{\infty} dy\, Q(y) \left(L_N\, e^{-\frac{|x-y|}{L_N}} - L_T\, e^{-\frac{|x-y|}{L_T}} \right) .
$$

Thus

$$\frac{n}{N_0} = \frac{\gamma}{4L_N L_T}\,\frac{2L_N L_T}{L_N^2 - L_T^2}\int_{-\infty}^{\infty} dy\; Q(y)\left(L_N\, e^{-\frac{|x-y|}{L_N}} - L_T\, e^{-\frac{|x-y|}{L_T}}\right)$$

$$+\ \frac{R_T L_N}{4L_T^3}\,\frac{2L_N L_T}{L_N^2 - L_T^2}\int_{-\infty}^{\infty} dy\; Q(y)\left(L_N\, e^{-\frac{|x-y|}{L_N}} - L_T\, e^{-\frac{|x-y|}{L_T}}\right)$$

$$-\ \frac{R_T L_N}{2L_T 2}\int_{-\infty}^{\infty} dy\; Q(y)\; e^{-\frac{|x-y|}{L_N}}\ ,$$

$$\frac{n}{N_0} = \frac{\gamma L_N L_T}{2\left(L_N^2 - L_T^2\right)}\int_{-\infty}^{\infty} dy\; Q(y)\left[\frac{e^{(-|x-y|/L_N)}}{L_T} - \frac{(-|x-y|/L_T)}{L_N}\right]\int_{-\infty}^{\infty} +\ dy\; Q(y)$$

$$\left\{\left[L_N\, e^{(-|x-y|/L_N)} - L_T\, e^{(-|x-y|/L_T)}\right]\left(\frac{R_T L_N^2}{2L_T^2}\cdot\frac{1}{L_N^2 - L_T^2}\right) - \frac{R_T L_N}{2L_T^2}\, e^{(-|x-y|/L_N)}\right\}.$$

The second term is equal to:

$$\int_{\infty}^{\infty} dy\; Q(y)\left\{\left[\frac{R_T L_N 3}{2L_T^2}\left(\frac{1}{L_N^2 - L_T^2}\right) - \frac{R_T L_N}{2L_T^2}\right] e^{-\frac{|x-y|}{L_N}} - \frac{R_T L_N 2}{2L_T}\cdot\frac{1}{L_N^2 - L_T^2}\; e^{-\frac{|x-y|}{L_T}}\right\}$$

$$= \frac{R_T L_N 2}{2\left(L_N^2 - L_T^2\right)}\int_{-\infty}^{\infty} dy\; Q(y)\left\{\left[\frac{L_N}{L_T^2} - \frac{\left(L_N^2 - L_T^2\right)}{R_T L_N{}^2}\,\frac{R_T L_N}{L_T^2}\right] e^{-\frac{|x-y|}{L_N}} - \frac{1}{L_T}\; e^{-\frac{|x-y|}{L_T}}\right\}.$$

And so we find the following:

$$\frac{L_N}{L_T^2} - \frac{\left(L_N^2 - L_T^2\right)}{R_T L_N^2} \; \frac{R_T L_N}{L_T^2} = \frac{L_N}{L_T^2} - \frac{\left(L_N^2 - L_T^2\right)}{L_N L_T^2}$$

$$= \frac{1}{L_T^2}\left[L_N - \frac{\left(L_N^2 - L_T^2\right)}{L_N} \right] = \frac{1}{L_N} \, .$$

Hence,

$$\frac{n}{N_0} = \frac{\gamma L_N L_T}{2\left(L_N^2 - L_T^2\right)} \int_{-\infty}^{\infty} dy \; Q(y) \left(\frac{e^{-\frac{|x-y|}{L_N}}}{L_T} - \frac{e^{-\frac{|x-y|}{L_T}}}{L_N} \right)$$

$$- \frac{R_T L_N^2}{2\left(L_N^2 - L_T^2\right)} \int_{-\infty}^{\infty} dy \; Q(y) \left(\frac{e^{-\frac{|x-y|}{L_N}}}{L_N} - \frac{e^{-\frac{|x-y|}{L_T}}}{L_T} \right) .$$

We note that

$$G\left(x, y, L_N, \; L_T\right) = \frac{\gamma L_N L_T}{2\left(L_N^2 - L_T^2\right)} \left(\frac{e^{\frac{-|x-y|}{L_N}}}{L_T} - \frac{e^{\frac{-|x-y|}{L_T}}}{L_N} \right)$$

$$\frac{- R_T L_N^2}{2\left(L_N^2 - L_T^2\right)} \left(\frac{e^{\frac{-|x-y|}{L_N}}}{L_N} - \frac{e^{\frac{-|x-y|}{L_T}}}{L_T} \right) ,$$

appears to be singular when $L_N = L_T$, which, according to Gurevich [1], occurs around 300 km. In fact, the expression has a *removable* singularity here as we shall now show. We let $l = L_N$ and $l_0 = L_T$, then:

$$\lim_{l \to l_o} G\left(x,y,l,l_o\right) = \lim_{l \to l_o} \frac{1}{2\left(l^2-l_o^2\right)} \left[\left(\gamma-R_T\right) e^{-\frac{|x-y|}{l}} - \left(\frac{\gamma l_o}{l} - \frac{R_T l}{l_o}\right) e^{-\frac{|x-y|}{l_o}}\right] .$$

In the above $G(x,y,l,l_o) = f(l)/g(l)$, where both $f(l)\to 0$ and $g(l)\to 0$ as $l\to l_o$. Hence, by L'Hospital's rule, $\lim_{l\to l_o} G(l) = \lim_{l\to l_o} f'(l)/g'(l)$, where

$$f(l) = \left(\gamma-R_T\right) e^{-\frac{|x-y|}{l}} - \left(\frac{\gamma l_o}{l} - \frac{R_T l}{l_o}\right) e^{-\frac{|x-y|}{l_o}} ,$$

$$f'(l) = \frac{|x-y|}{l^2} \left(\gamma - R_T\right) e^{-\frac{|x-y|}{l}} + \left(\frac{\gamma l_o}{l^2} - \frac{R_T}{l_o}\right) e^{-\frac{|x-y|}{l_o}} ,$$

$$g(l) = \frac{2\left(l^2-l_o^2\right)}{l} = 2l - \frac{2l_o^2}{l} ,$$

$$g'(l) = 2 + \frac{2l_o^2}{l^2} .$$

Therefore, we find that

$$\lim_{l\to l_o} \frac{f'(l)}{g'(l)} = \frac{1}{4}\left[\frac{|x-y|}{l_o^2} \left(\gamma-R_T\right) e^{-\frac{|x-y|}{l_o}} + \frac{\left(\gamma+R_T\right)}{l_o} e^{-\frac{|x-y|}{l_o}}\right]$$

$$= \frac{1}{4l_o} e^{-\frac{|x-y|}{l_o}} \left[|x-y| \frac{\left(\gamma-R_T\right)}{l_o} + \left(\gamma+R_T\right)\right] .$$

and,

$$\lim_{L_N \to L_T} G\left(x, y, L_N, L_T\right) = \frac{1}{4L_N} e^{-\frac{|x-y|}{L_N}} \left[\frac{|x-y|}{L_N}\left(\gamma - R_T\right) + \left(\gamma + R_T\right)\right] .$$

and where $L_N = L_T$ (at 300 km),

$$\frac{n}{N_o} = \frac{1}{4L_N} \int_{-\infty}^{\infty} Q(y) \left[\frac{|x-y|}{L_N}\left(\gamma - R_T\right) + \left(\gamma + R_T\right)\right] e^{-\frac{|x-y|}{L_N}} dy ,$$

which is well-defined.

Appendix D

SOLUTION FOR SQUARE HEATED REGION

We consider an idealized joule-heating source,

$$E(x,t) = \begin{cases} E_o & -L \le x \le L \text{ and } o \le t \le t_1 , \\ 0 & \text{elsewhere} , \end{cases}$$

and integrate the expression

$$\frac{\Delta T(x,t)}{T_{eo}} = \frac{1}{\tau_T} \int^t ds \int_{-\infty}^{\infty} \frac{\exp\left[- \frac{(t - s)}{\tau_T} - \frac{(x - \xi^2)}{4\kappa(t - s)}\right]}{\sqrt{4\pi\kappa(t - s)}} \frac{E^2}{E_p^2} (\xi, s)\, ds ,$$

where $\kappa = L_T^2/\tau_T$.

Thus we find:

$$\frac{\Delta T(x,t)}{T_{eo}} = \frac{E_o^2}{E_p^2} \frac{1}{\tau_T} \int_o^T \frac{\exp\left[- \frac{(t - s)}{\tau_T}\right]}{\sqrt{4\pi\kappa(t - s)}} \int_{-L}^{L} \exp\left[- \frac{(x - \xi^2)}{4\kappa(t - s)}\right] d\xi\, ds ,$$

where

$$T = \begin{cases} t & \text{if } t \le t_1 , \\ t_1 & \text{if } t > t_1 . \end{cases}$$

We shall see that, in the event the source is turned off at a finite time ($t_1 < +\infty$), the perturbation will naturally decay to zero. Now,

$$\int_{-L}^{L} \exp\left[-\frac{(x-\xi^2)}{4\kappa(t-s)}\right] d\xi = \int_{-(L+x)}^{L-x} \exp\left[-\frac{\mu 2}{4\kappa(t-s)}\right] d\mu$$

$$= \frac{\sqrt{4\pi\kappa(t-s)}}{2}\left[\operatorname{erf}\left(\frac{L-x}{\sqrt{4\kappa(t-s)}}\right) + \operatorname{erf}\left(\frac{L+x}{\sqrt{4\kappa(t-s)}}\right)\right],$$

where

$$\operatorname{erf}(Z) = \frac{2}{\sqrt{\pi}} \int_{o}^{Z} e^{-t^2}\, dt \;.$$

Hence, we find:

$$\frac{\Delta T_e}{T_{eo}} = \frac{E_o^2}{E_p^2}\,\frac{1}{2\cdot\tau_T}\int_o^T \exp\left(-\frac{(t-s)}{\tau_T}\right)\left[\operatorname{erf}\left(\frac{L-x}{\sqrt{4\kappa(t-s)}}\right) + \operatorname{erf}\left(\frac{L+x}{\sqrt{4\kappa(t-s)}}\right)\right] ds \;.$$

We then use the following, from Abramowitz and Stegun [15]:

$$\int e^{av} \operatorname{erf}\sqrt{b/v}\, dv = \frac{1}{a}\left\{e^{av} \operatorname{erf}\sqrt{b/v} + \frac{1}{2}\, e^{[av-(b/v)]}\right.$$

$$\left.\cdot\left[w(\sqrt{av} + i\sqrt{b/v}) + w(-\sqrt{av} + i\sqrt{b/v})\right]\right\},$$

where $w(z) = e^{-z^2}\operatorname{erfc}(-iz)$,

$\operatorname{erfc}(z) = 1 - \operatorname{erf}(z)$,

$i = \sqrt{-1}$).

Hence, we have the following:

$$\int_o^T \exp\left(-\frac{(t-s)}{\tau_T}\right) \text{erf}\left(\frac{L \pm x}{\sqrt{4\kappa(t-s)}}\right) ds = -\int_t^{t-T} \exp\left(\frac{-u}{\tau_T}\right) \text{erf}\left(\frac{L \pm x}{\sqrt{4\kappa u}}\right) du$$

$$= \tau_T \left\{ \exp\left(\frac{-u}{\tau_T}\right) \text{erf}\left(\frac{L \pm x}{\sqrt{4\kappa(t-s)}}\right) + \frac{1}{2}\exp\left[-\frac{u}{\tau_T} - \frac{(L \pm x)^2}{4\kappa u}\right]\right.$$

$$\left. \cdot \left[w\left(\sqrt{-u/\tau_T} + i\,\frac{|L \pm x|}{\sqrt{4\kappa u}}\right) + w\left(-\sqrt{-u/t} + i\,\frac{|L \pm x|}{\sqrt{4\kappa u}}\right)\right]_{u=t}^{u=t-T} \right. .$$

Note, since $u > o$, $\pm\sqrt{-u/\tau_T} = \pm i\,\sqrt{u/\tau_T}$.

Now we find that

$$w\left[i\left(\pm\sqrt{\frac{u}{\tau_T}} + \frac{|L \pm x|}{\sqrt{4\kappa u}}\right)\right] = \exp\left[\frac{u}{\tau_T} + \frac{(L \pm x)^2}{4\kappa u} \pm \frac{|L \pm x|}{\sqrt{\kappa\tau_T}}\right]$$

$$\cdot\ \text{erfc}\left(\pm\sqrt{\frac{u}{\tau_T}} + \frac{|L \pm x|}{\sqrt{4\kappa u}}\right) ,$$

therefore,

$$\frac{\Delta T_e}{T_{eo}} = \frac{E_o^2}{2E_p^2}\left\{\exp\left(-\frac{u}{\tau_T}\right)\left[\text{erf}\left(\frac{L-x}{\sqrt{4\kappa u}}\right) + \text{erf}\left(\frac{L+x}{\sqrt{4\kappa u}}\right)\right]\right.$$

$$+ \frac{1}{2}\exp\left(\frac{|L-x|}{\sqrt{\kappa\tau}}\right)\text{erfc}\left(\sqrt{\frac{u}{\tau_T}} + \frac{|L-x|}{\sqrt{4\kappa u}}\right) + \frac{1}{2}\exp\left(-\frac{|L-x|}{\sqrt{\kappa\tau}_T}\right)$$

$$\cdot\ \text{erfc}\left(-\sqrt{\frac{u}{\tau_T}} + \frac{|L-x|}{\sqrt{4\kappa u}}\right) + \frac{1}{2}\exp\left(\frac{|L\pm x|}{\sqrt{\kappa\tau_T}}\right)\text{erfc}\left(\sqrt{\frac{u}{\tau_T}} + \frac{|L-x|}{\sqrt{4\kappa u}}\right)$$

$$\left. + \frac{1}{2}\exp\left(-\frac{|L\pm x|}{\sqrt{\kappa\tau_T}}\right)\text{erfc}\left(\sqrt{\frac{u}{\tau_T}} + \frac{|L+x|}{\sqrt{4\kappa u}}\right)\right\}_{u=t}^{u=t-T} . \qquad (D.1)$$

If $t \leq t_1$ $T = t$, and observing that $\lim_{u\to\infty} \text{erfc}(u) = o$ and $\lim_{u\to\infty} \text{erf}(u) = 1$, we find that

$$\frac{\Delta T_e}{T_{eo}} = \frac{E_o^2}{E_p^2}\left\{1 - \frac{1}{2}\exp\left(-\frac{t}{\tau_T}\right)\left[\text{erf}\left(\frac{L-x}{\sqrt{4\kappa t}}\right) + \left(\frac{L+x}{\sqrt{4\kappa t}}\right)\right]\right\}$$

$$- \frac{E_o^2}{4E_p^2}\left[\exp\left(\frac{|L-x|}{\kappa\tau_T}\right)\text{erfc}\left(\sqrt{\frac{t}{\tau_T}} + \frac{|L-x|}{\sqrt{4\kappa t}}\right) + \exp\left(-\frac{|L-x|}{\kappa\tau_T}\right)\right.$$

$$\cdot\ \text{erfc}\left(-\sqrt{\frac{t}{\tau_T}} + \frac{|L-x|}{\sqrt{4\kappa t}}\right) + \exp\left(\frac{|L+x|}{\kappa\tau_T}\right)\text{erfc}\left(\sqrt{\frac{t}{\tau_T}} + \frac{|L+x|}{\sqrt{4\kappa t}}\right)$$

$$\left. + \exp\left(-\frac{|L+x|}{\kappa\tau_T}\right)\text{erfc}\left(-\sqrt{\frac{t}{\tau_T}} + \frac{|L+x|}{\sqrt{4\kappa t}}\right)\right] .$$

If the heat source is never turned off, that is, taking $t_1 = +\infty$, we have:

$$\lim_{t\to\infty} \frac{\Delta T_E}{T_{eo}} = \frac{E_o^2}{E_p^2} \left\{ 1 - \frac{1}{2} \left[\exp\left(- \frac{|L - x|}{\sqrt{\kappa \tau_T}} \right) + \exp\left(- \frac{|L + x|}{\sqrt{\kappa \tau_T}} \right) \right] \right\} ,$$

since

$$\lim_{t\to\infty} \operatorname{erfc}\left(- \sqrt{\frac{t}{\tau_T}} + \frac{|L \pm x|}{\sqrt{4\kappa t}} \right) = 2 .$$

This is in agreement, of course, with the steady-state solution obtained by using the formula in Appendix B:

$$\frac{\Delta T_e}{T_{eo}} = \frac{1}{2\sqrt{\kappa \tau_T}} \int_{-\infty}^{\infty} Q(\xi) \exp\left(- \frac{|x - \xi|}{\sqrt{\kappa \tau_T}} \right) d\xi ,$$

where

$$Q(x) = \begin{cases} E_o^2/E_p^2 & |x| \le L , \\ 0 & |x| > L . \end{cases}$$

Moreover, if $t_1 < \infty$ and we take $T = t_1$ in Eq. (D.1), the limit is clearly zero.